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A perfect duality between p-adic Banach spaces and compactoids

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ABSTRACT

In the set up of Non-Archimedean Functional Analysis, Banach spaces as well as compactoids play a fundamental role, although casted for different parts. The more surprising it is, that yet both concepts turn out to be closely related as will be revealed in Part One. In fact we shall prove that (roughly speaking) to any mathematical statement about compactoids there exists an equivalent dual statement formulated in terms of Banach spaces, and conversely. This is done by establishing an anti-equivalence between categories (Theorem 4.6). In Part Two we consider several consequences.

Remark. The report [7] can be viewed as a forerunner of the present paper whereas in [11] and [12] some remaining details are being worked out.

Part One

Banach spaces versus compactoids, an anti-equivalence

§1. PROLOGUE

(For notations and terminology see the next section.) Recall [13] that a convex subset C of a Hausdorff locally convex space over a (spherically complete) non-archimedean valued field K is said to be *c-compact* if every collection of relatively closed convex subsets of C with the finite intersection property has a non-void intersection. This notion suitably replaces the ordinary concept 'convex-compact' in case K is not locally compact; in fact, certain convexified versions of compactness properties hold for c-compact sets C as well. For example, C is complete; if T is a continuous linear map then TC is c-compact, if in addition T is injective then T is a homeomorphism $C \to TC$. Within this context it therefore is natural to ask whether TC always carries the quotient topology induced by T. As a test case consider the following situation. Let K be spherically complete, let A be an absolutely convex bounded c-compact (= an absolutely convex closed compactoid) set in c_0, let $T \in \mathcal{L}(c_0)$. It is not hard to see that, by metrizability and the fact that A is an additive topological group, the following assertions are equivalent.

(α) The (norm) topology on TA equals the quotient topology induced by T.

(β) $T : A \to TA$ is an open mapping.

(γ) For each sequence $y_1, y_2, \ldots$ in TA tending to 0 there exists a sequence $x_1, x_2, \ldots$ in A tending to 0 such that $Tx_n = y_n$ for each n.

It is known that (α)–(γ) hold if the valuation of K is discrete ([10] § 3) or if T is injective ([1], [4], [5]). The following startling example shows that (α)–(γ) are false in general thus providing a negative answer to the problem stated in [10] § 3. It was first obtained as a by-product of the main theory of this paper (see Theorem 9.5) but here it is considered to be shorn of these ornaments.

Example 1.1. Let K be spherically complete and let the valuation be dense. Then there exist a closed absolutely convex compactoid A in c_0 and a $T \in \mathcal{L}(c_0)$ such that its restriction $A \to TA$ is not an open mapping.

Proof. Choose $b_1, b_2, \ldots \in K$ with $0 < |b_n| < 1$ for each n and $\prod_n |b_n| > 0$, choose $\rho \in K$, $0 < |\rho| < 1$. The formula

$$T\left(\sum_{n=1}^{\infty} \lambda_n e_n \right) = \sum_{n=1}^{\infty} (\lambda_n - b_n \lambda_{n+1} \rho^{-1}) e_n$$

where $e_1, e_2, \ldots$ is the standard base of c_0 and $\lambda_n \in K$, $\lambda_n \to 0$, defines a map $T \in \mathcal{L}(c_0)$. Setting $y_n := \rho^n e_n$ we have $y_n \to 0$ so that

$$A := \overline{\text{co}}\{y_1, y_2, \ldots\}$$

is a closed compactoid. We first show that $y_1, y_2, \ldots$ are in TA. In fact, we have $y_n = Tz_n$ where

$$z_n := y_n + b_{n-1} y_{n-1} + b_{n-2} b_{n-1} y_{n-1} + \cdots + b_1 b_2 \ldots b_{n-1} y_1 \in A.$$

Also observe that Ker $T = Ka$ where $a := (1, \rho b_1^{-1}, \rho^2 b_1^{-1} b_2^{-1}, \ldots) \in c_0$. Now let $x_1, x_2, \ldots$ be any sequence in A with $Tx_n = y_n$ for each n. Then $x_n = z_n - \mu_n a$ where $\mu_n \in K$. Then $\mu_n a \in A$ implying $|\mu_n| \leq \prod |b_i|$. The first coefficient of x_n in the expansion with respect to $y_1, y_2, \ldots$ equals $b_1 b_2 \ldots b_{n-1} - \mu_n$ which does not tend to 0 since $|\mu_n| < |b_1 \ldots b_{n-1}|$. It follows that $x_1, x_2, \ldots$ cannot tend to 0 and (γ) above is not true. □

Remark 1. The above proof works also when K is not spherically complete although in this case the result is less spectacular as continuous linear images of A need not even be closed! ([2], 6.28).

Remark 2. Observe that A of above is edged in the sense of 2.2 below.

§2. TERMINOLOGY

In this section we collect definitions, notations, conventions, ... needed in this paper. For terms that remain unexplained we refer to [3], [6].

2.1. Throughout K is a non-archimedean valued field that is complete with respect to its non-trivial valuation $|\ |$. We set $B_K := \{\lambda \in K : |\lambda| \leq 1\}$ and $B_K^- := \{\lambda \in K : |\lambda| < 1\}$.

2.2. A subset A of a K-vector space E is *absolutely convex* if it is a B_K-submodule of E. For a subset X of E we denote by co X its absolutely convex hull, by $[X]$ its linear span. For an absolutely convex $A \subset E$ the formula

$$p_A(x) = \inf\{|\lambda| : \lambda \in K, x \in \lambda A\}$$

defines a (non-archimedean) seminorm p_A on $[A]$ called *the Minkowski function of A*. We define $A^e := A$ if the valuation of K is discrete and $A^e := \bigcap\{\lambda A : \lambda \in K, |\lambda| > 1\}$ if the valuation of K is dense. A is *edged* if $A = A^e$ or, equivalently, if $A = \{x \in [A] : p_A(x) \leq 1\}$.

A seminorm p on E is *polar* if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$ when E^* is the space of all linear functions $E \to K$.

2.3. Let $E = (E, \|\ \|)$ be a normed space over K. (Throughout norms are assumed to be non-archimedean.) For $a \in E$, $r > 0$ we write $B_E(a, r) := \{x \in E : \|x - a\| \leq r\}$ (the 'closed' ball) and $B_E^-(a, r) := \{x \in E : \|x - a\| < r\}$ (the 'open' ball). The 'closed' unit ball $B_E(0, 1)$ is sometimes denoted B_E; similarly $B_E^- := B_E^-(0, 1)$. For $a \in E, X \subset E$ we set $\operatorname{dist}(a, X) := \inf\{\|a - x\| : x \in X\}$.

Let E, F be K-Banach spaces. Then $\mathcal{L}(E, F) := \{T : E \to F : T$ linear and continuous$\}$ is a K-Banach space under the norm

$$T \mapsto \|T\| := \inf\{M \geq 0 : \|Tx\| \leq M\|x\| \text{ for all } x \in E\}.$$

As usual we write $\mathcal{L}(E) := \mathcal{L}(E, E)$ and $E' := \mathcal{L}(E, K)$.

Let D be a closed subspace of E, let $\pi : E \to E/D$ be the canonical map. The *quotient norm* on E/D is defined by the formula $\|\pi(x)\| = \operatorname{dist}(x, D)$. Then π maps $B_E^-(0, r)$ onto $B_{E/D}^-(0, r)$ for each $r > 0$ and is called a *quotient map*.

For each $T \in \mathcal{L}(E, F)$ its *adjoint* $T' \in \mathcal{L}(F', E')$ is defined by $T'(f) = f \circ T$ $(f \in F')$. It is easily seen that $\|T'\| \leq \|T\|$ and that the diagram

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "j_E"'] & F \arrow[d, "j_F"] \\ E'' \arrow[r, "T''"'] & F'' \end{tikzcd}$$

commutes. Here, of course, j_E, j_F are the natural maps.

A normed space $(E, \| \ \|)$ is *normpolar* if $\| \ \|$ is polar (see 2.2). If the valuation of K is discrete E is normpolar if and only if $\|E\| \subset |K|$. If the valuation of K is dense E is normpolar if and only if for each finite dimensional subspace D, for each $f \in D'$, for each $\varepsilon > 0$ there exists an extension $\tilde{f} \in E'$ of f for which $\|\tilde{f}\| \leq (1+\varepsilon)\|f\|$. Thus, if in addition K is spherically complete, every K-normed space is normpolar. For normpolar spaces E, F and $T \in \mathcal{L}(E, F)$ we have $\|T\| = \sup\{\|Tx\| : x \in B_E\}$ and $\|T'\| = \|T\|$. Also the canonical map $j_E : E \to E''$ is isometrical. E is *reflexive* if j_E is an isometrical bijection.

2.4. A locally convex space E over K is *polar* if there exists a base of continuous polar seminorms. Set $E' := \{f \in E^* : f \text{ is continuous}\}$. The *weak topology* $w = \sigma(E, E')$ is the weakest topology on E for which all $f \in E'$ are continuous. Similarly, the *weak star topology* $w' = \sigma(E', E)$ is the weakest topology on E' for which all evaluations $f \mapsto f(x)$ $(x \in E)$ are continuous.

A subset X of a locally convex space E is (a) *compactoid* if for every zero neighbourhood U in E there exists a finite set $F \subset E$ such that $X \subset U + \text{co } F$. Then the space $[X]$ is of countable type ([6], 4.3). Recall that, on compactoids in a polar locally convex space, the weak topology and the initial topology coincide ([6], 5.12). The closure of a set $Y \subset E$ is $\overline{Y}$. Instead of $\overline{\text{co } Y}$ we write $\overline{\text{co}}\ Y$.

2.5. We shall adopt the convention to say that a map $f : X \to Y$, where X, Y are topological spaces, is a *homeomorphism into* if f maps X homeomorphically onto $f(X)$, equipped with the relative topology.

§ 3. THE WEAK STAR UNIT BALL OF E'

Proposition 3.1 (p-adic Alaoglu theorem). *Let E be a K-Banach space. Then the w'-unit ball $B_{E'}$ is an absolutely convex, edged, complete compactoid.*

Proof. The formula $\phi(f) = (f(x))_{x \in B_E}$ defines a B_K-module homomorphism $\phi : B_{E'} \to B_K^{B_E}$ which is a homeomorphism into. Thus, $B_{E'}$ is isomorphic to a subset of a compactoid hence is one itself. The proofs of w'-completeness, absolute convexity and edgedness are straightforward. □

The *bounded weak star topology* bw' on a K-Banach space E is the strongest locally convex topology on E' coinciding with w' on bounded subsets of E'. In other words, a (non-archimedean) seminorm p on E' is bw'-continuous if and only if $p \mid B_{E'}$ is w'-continuous.

Proposition 3.2. *Let E be a K-Banach space, let $j_E : E \to E''$ be the canonical map. Then we have*

(i) *The dual of (E', w') is $j_E(E)$.*
(ii) *The dual of (E', bw') is the norm closure of $j_E(E)$ in E''.*

Proof. See [9], 3.3.

Corollary 3.3. *For a normpolar space E the dual of (E', bw') is $j_E(E)$.*

Proposition 3.4 (p-adic Goldstine theorem). *Let E be a normpolar K-Banach space. Then the map j_E is a homeomorphism of (E, w) into (E'', w') whose image is dense. We even have $(\overline{j_E(B_E)}^{w'})^e = B_{E''}$.*

Proof. We only prove the last statement. Since j_E is an isometry we have $j_E(B_E) \subset B_{E''}$ and the p-adic Alaoglu theorem yields that even $A := (\overline{j_E(B_E)}^{w'})^e \subset B_{E''}$. A is closed and edged hence ([6], 4.8) a polar set in the topology $w' = \sigma(E'', E')$. So if $\theta \in B_{E''}$, $\theta \notin A$ we could find an element Ω in the dual of (E'', w') such that $|\Omega(\theta)| > 1$ and $|\Omega| \leq 1$ on A. But by Proposition 3.2(i) Ω has the form $\theta \mapsto \theta(f)$ for some $f \in E'$. We then have $|\theta(f)| > 1$ and $|j_E(B_E)(f)| \leq 1$ i.e. $|f| \leq 1$ on B_E which, by polarity, just means $\|f\| \leq 1$. But then $|\theta(f)| \leq \|\theta\| \, \|f\| \leq 1$, a contradiction. □

Proposition 3.5. *Let E, F be normpolar K-Banach spaces, let $S \in \mathcal{L}(F', E')$, $\|S\| \leq 1$. Then the following are equivalent.*

(α) *S is the adjoint of a $T \in \mathcal{L}(E, F)$, $\|T\| \leq 1$.*
(β) *S is continuous $(F', w') \to (E', w')$.*
(γ) *$S \mid B_{F'}$ is continuous $(B_{F'}, w') \to (B_{E'}, w')$.*
(δ) *S is continuous $(F', bw') \to (E', bw')$.*

Proof. The implications $(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma)$ are obvious. Suppose (γ). If p is a bw'-continuous seminorm on E' then $p \circ S$ is w'-continuous on $B_{F'}$ so by definition $p \circ S$ is bw'-continuous and (δ) follows. Finally we prove $(\delta) \Rightarrow (\alpha)$. For each $x \in E$ we have $S'(j_E(x)) = j_E(x) \circ S \in (F', bw')'$ which equals $j_F(F)$ by Corollary 3.3. We see that S' maps $j_E(E)$ into $j_F(F)$ so by polarity there exists a unique map $T : E \to F$ (which is linear and continuous) making the diagram

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ {\scriptstyle j_E}\downarrow & & \downarrow{\scriptstyle j_F} \\ E'' & \xrightarrow[S']{} & F'' \end{array}$$

commute. We see that $T'' = S'$ on $j_E(E)$, hence on E'' by w'-continuity and Proposition 3.4. Hence $(T' - S)' = 0$, so $T' = S$ and we have (α) as also $\|T\| = \|T'\| = \|S\| \leq 1$. □

§4. THE ANTI-EQUIVALENCE

In this section we prove the core of this paper, namely that the theory of normpolar Banach spaces is equivalent to the theory of complete edged compactoids in locally convex spaces. First let us describe in 4.1 and 4.2 the categories involved more precisely.

4.1. The category $\mathcal{B}_K$ of Banach spaces. We shall denote by $\mathcal{B}_K$ the category of the normpolar K-Banach spaces where for $E, F \in \mathcal{B}_K$ we set

$$\operatorname{Hom}(E,F) := \{T \in \mathcal{L}(E,F) : \|T\| \le 1\}.$$

4.2. The category $\mathcal{C}_K$ of compactoids. This takes a little bit of preparation. To define the objects of $\mathcal{C}_K$ properly we shall 'free' an absolutely convex set from the space it is embedded in, which can be done as follows. Let A be a module over the ring B_K, let τ be a topology on A. We call $A = (A, \tau)$ a *topological module* if the module operations are continuous. Any absolutely convex subset of a topological vector space over K is, with the inherited topology, a topological module. Conversely, we will say that a topological module (A, τ) is *embeddable* if there exists a Hausdorff locally convex space E over K and a B_K-module homomorphism $i : A \to E$ which is also a homeomorphism into. We say that A is *(a) compactoid* if $i(A)$ is (a) compactoid, *edged* if $i(A)$ is edged. (It is not hard to see that these definitions do not depend on the particular choices of E and i; one even can formulate intrinsic equivalent definitions as follows. A topological B_K-module (A, τ) is a compactoid if for each τ-neighbourhood U of 0 and each $\lambda \in B_K^-$ there exists a finitely generated submodule F of A such that $\lambda A \subset U + F$. A B_K-module A is edged if each homomorphism $B_K^- \to A$ can be extended to a homomorphism $B_K \to A$.) Now we are able to introduce the second category.

We shall denote by $\mathcal{C}_K$ the category of the embeddable, edged, complete compactoid topological B_K-modules where for $A, B \in \mathcal{C}_K$ we set

$$\operatorname{Hom}(A,B) := \{\varphi : A \to B : \varphi \text{ is a continuous homomorphism}\}.$$

Note. For a more sophisticated definition of $\mathcal{C}_K$ avoiding embeddings in locally convex spaces, see [11]. For practical reasons in this paper we prefer the introduction of $\mathcal{C}_K$ as given above.

4.3. The functor $\mathcal{B}_K \to \mathcal{C}_K$. To each $E \in \mathcal{B}_K$ we assign the weak star unit ball $(B_{E'}, w')$. We shall often write $B_{E'}$ rather than $(B_{E'}, w')$. By the p-adic Alaoglu theorem 3.1 this $B_{E'}$ is in $\mathcal{C}_K$.

Now let $E, F \in \mathcal{B}_K$ and $T \in \operatorname{Hom}(E,F)$. By normpolarity for the adjoint $T' : F' \to E'$ we have $\|T'\| = \|T\|$ so that $T' \in \operatorname{Hom}(F', E')$. The restriction T^d of T' to the unit ball $B_{F'}$ maps into $B_{E'}$. It is easily seen (Proposition 3.5 if you wish) that $T^d : B_{F'} \to B_{E'}$ is an element of $\operatorname{Hom}(B_{F'}, B_{E'})$. One verifies without pain that

$$T \mapsto T^d \quad (T \in \operatorname{Hom}(E,F),\ E, F \in \mathcal{B}_K)$$

defines a contravariant functor $\mathcal{B}_K \to \mathcal{C}_K$.

Now let's get busy.

Proposition 4.4. *Let $E, F \in \mathcal{B}_K$. The map $T \mapsto T^d$ is a bijection* $\operatorname{Hom}(E,F) \to \operatorname{Hom}(B_{F'}, B_{E'})$.

Proof. We only need to inspect surjectivity. If $\varphi \in \mathrm{Hom}(B_{F'}, B_{E'})$ then its unique linear extension $S : F' \to E'$ has norm ≤ 1 and satisfies (γ), hence (α) of Proposition 3.5. Thus $\varphi = T^d$ for some $T \in \mathrm{Hom}(E, F)$. $\square$

Proposition 4.5. *Every $A \in \mathcal{C}_K$ is isomorphic to $B_{E'}$ for some suitably chosen $E \in \mathcal{B}_K$.*

Proof. 1. Let τ be the topology of A. We may assume that A is an absolutely convex edged complete compactoid subset of some Hausdorff locally convex space (X, ν). Define a locally convex topology τ_1 on $Y := [A]$ by declaring a seminorm p on Y to be τ_1-continuous as soon as $p \mid A$ is τ-continuous. Then τ_1 is stronger than $\nu \mid Y$ but $\tau_1 = \nu = \tau$ on A. We also know that (Y, τ_1) is of countable type (2.4), hence a polar space ([6], 4.4).

2. Now let E be the linear space $(Y, \tau_1)'$ equipped with the norm $f \mapsto \|f\| := \sup\{|f(x)| : x \in A\}$. Then, indeed, $\| \ \|$ is a polar norm. To prove completeness, let $f_1, f_2, \ldots$ be a Cauchy sequence in E. Then for each $y \in Y$

$$f(y) := \lim_{n \to \infty} f_n(y)$$

exists and is a linear function on Y. As $f_n \to f$ uniformly on A we have that $f \mid A$ is continuous and so is its absolute value. By the very construction of τ_1 the seminorm $|f|$ is τ_1-continuous. It follows that $f \in E$ and that $\|f - f_n\| \to 0$. We conclude that $E \in \mathcal{B}_K$.

3. Finally we show that (A, τ) is isomorphic to $(B_{E'}, w')$. Consider the map $j : Y \to E'$ defined by the formula $j(y)(f) = f(y)$ $(y \in Y, f \in E)$. This j is clearly a homeomorphism of (Y, w) into (E', w'). Since Y is a polar space we have $w = \tau$ on A (2.4). Thus j maps (A, τ) homeomorphically into $(B_{E'}, w')$ so that $j(A)$ is edged and w'-complete hence w'-closed. If we had an $\Omega \in B_{E'} \setminus j(A)$ there would exist, by strong polarity of the space (E', w'), a w'-continuous linear function on E' separating $j(A)$ and $\{\Omega\}$ i.e. we would have an $f \in E$ with $|f| \leq 1$ on A (which precisely means $\|f\| \leq 1$) and $|\Omega(f)| > 1$. But then also $|\Omega(f)| \leq \|\Omega\| \, \|f\| \leq 1$, a contradiction. It follows that $j \mid A$ is surjective. Thus, $j \mid A$ is an isomorphism between the topological B_K-modules A and $B_{E'}$. $\square$

Remark 1. Part 3 of the above proof can be shortened by using the p-adic Mackey theorem ([6], 7.4).

Remark 2. We have the following immediate corollary to Proposition 4.5.

Let A be a complete absolutely convex compactoid in a Hausdorff locally convex space over K. Then $([A], p_A)$ is the dual of some K-Banach space. (For p_A see 2.2.)

Combination of the previous two propositions yields the following.

Theorem 4.6. *The categories $\mathcal{B}_K$ (of all normpolar K-Banach spaces) and $\mathcal{C}_K$ (of* *all embeddable absolutely convex complete edged compactoids) are anti-equivalent by means of the functor $\mathcal{B}_K \to \mathcal{C}_K$ given by*

$$E \mapsto B_{E'} \quad (E \in \mathcal{B}_K)$$
$$T \mapsto T^d \quad (T \in \mathrm{Hom}(E,F),\ E,F \in \mathcal{B}_K)$$

where $B_{E'}$ carries the restriction of the w'-topology of E' and where $T^d := T' \mid B_{F'} \in \mathrm{Hom}(B_{F'}, B_{E'})$.

Remark 3. Theorem 4.6 can be proved in a much more direct way in case the valuation of K is discrete, by using that (i) each $E \in \mathcal{B}_K$ has an orthonormal base ([2], 5.16) and that (ii) each $A \in \mathcal{C}_K$ is isomorphic to some power of B_K ([5], Theorem 16).

Remark 4. Given an $A \in \mathcal{C}_K$ we can quickly construct its corresponding object E in $\mathcal{B}_K$ as follows. Set $E := \mathrm{Hom}(A, K)$ (with that we mean, of course, the set of all continuous B_K-module homomorphisms $A \to K$); it is in a natural way a linear space and $f \to \|f\|_A := \sup\{|f(x)| : x \in A\}$ makes E into a normpolar Banach space. From the proof of Proposition 4.5 it follows directly that the w'-unit ball of E' is isomorphic to A.

4.7. Notation. Let $E, F \in \mathcal{B}_K$ and let $\varphi \in \mathrm{Hom}(B_{F'}, B_{E'})$. The unique $T \in \mathrm{Hom}(E,F)$ for which $T^d = \varphi$ is denoted φ^d. Then obviously $T^{dd} = T$ and $\varphi^{dd} = \varphi$ for all $T \in \mathrm{Hom}(E,F)$ and $\varphi \in \mathrm{Hom}(B_{F'}, B_{E'})$.

Part Two

Banach spaces versus compactoids – Applications

The fact that each K-Banach space is a quotient of two spaces with an orthonormal base leads to the Structure Theorem 5.2 for compactoids. Further, we will characterize those $B_{E'}$ for which E is of countable type (6.1), or has an orthogonal base (8.1), or is reflexive (7.1). Finally, in § 9 we will discuss convexified forms of compact-like properties. Theorem 9.5 provides a complete answer to the problem on openness of surjections indicated in § 1.

§ 5. DECOMPOSITIONS IN $\mathcal{B}_K$ AND $\mathcal{C}_K$

For $E, F \in \mathcal{B}_K$ and $T \in \mathrm{Hom}(E,F)$ we have an obvious decomposition

$$\begin{tikzcd} E \arrow[rr] \arrow[dr, "T_1"'] & & F \\ & \overline{TE} \arrow[ur, "T_2"'] & \end{tikzcd}$$

into a map $T_1 \in \mathrm{Hom}(E, \overline{TE})$ whose image is dense and an isometry $T_2 \in \mathrm{Hom}(\overline{TE}, F)$. (We have avoided the usual further decomposition of T_1 into a quotient map $E \to E/\mathrm{Ker}\ T$ and a dense injection $E/\mathrm{Ker}\ T \to \overline{TE}$ because $E/\mathrm{Ker}\ T$ is, in general, not in $\mathcal{B}_K$.) We now look into its dual in $\mathcal{C}_K$. Recall that T^d is the restriction of $T' : F' \to E'$ to the weak star unit ball $B_{F'}$.

Theorem 5.1. *Let $E, F \in \mathcal{B}_K$, let $T \in \operatorname{Hom}(E, F)$. Then*

(i) *TE is norm dense in F if and only if T^d is a (w'-) homeomorphism of $B_{F'}$ into $B_{E'}$,*

(ii) *T is an isometry if and only if $(\overline{T^d(B_{F'})})^e = B_{E'}$, (the bar denoting the w'-closure).*

Proof. (i) Suppose TE is norm dense in F and let $i \mapsto g_i$ be a net in $B_{F'}$ such that $T^d g_i \xrightarrow{w'} 0$; we prove that $g_i \xrightarrow{w'} 0$. We have $g_i \circ T \xrightarrow{w'} 0$ i.e., $g_i \to 0$ pointwise on TE. Let $y \in F$. By assumption there exist $y_1, y_2, \ldots \in TE$ such that $\|y - y_n\| \to 0$. From $|g_i(y)| \leq \|g_i\| \, \|y - y_n\| \vee |g_i(y_n)| \leq \|y - y_n\| \vee |g_i(y_n)|$ for all i, n one infers easily that $g_i(y) \to 0$. Hence $g_i \xrightarrow{w'} 0$.

Conversely, let T^d be a homeomorphism into and suppose we had an $y \in F \setminus \overline{TE}$. Then y is s-orthogonal to $\overline{TE}$ for some $s > 0$. Choose a $\xi \in K$, $0 < |\xi| \leq \frac{1}{2}s\|a\|$. For each finite-dimensional subspace D of TE the map

$$\lambda y + d \mapsto \lambda \xi \quad (\lambda \in K,\ d \in D)$$

has norm $\leq \frac{1}{2}$ on $Ky + D$ and, by normpolarity, can be extended to a $g_D \in B_{F'}$. These g_D form a net in a natural way. We have $T^d(g_D) = g_D \circ T \xrightarrow{w'} 0$, so, by assumption, $g_D \xrightarrow{w'} 0$ which is a contradiction as $|g_D(y)| = |\xi| > 0$ for all D.

(ii) Suppose T is an isometry. It is not hard to see that $(\overline{T^d(B_{F'})})^e \subset B_{E'}$. If this inclusion were strict we would have an $f \in B_{E'}$ that can be separated from $(\overline{T^d(B_{F'})})^e$ by a w'-continuous linear function i.e. there is an $x \in E$ with $|f(x)| > 1$ and $|T^d(B_{F'})(x)| \leq 1$. The first inequality entails $1 < |f(x)| \leq \|f\| \, \|x\| \leq \|x\|$ whereas the second one yields $|g(Tx)| \leq 1$ for all $g \in B_{F'}$ i.e., $\|Tx\| \leq 1$. Hence $\|Tx\| < \|x\|$, a contradiction.

Conversely, suppose $(\overline{T^d(B_{F'})})^e = B_E{}'$. Let $x \in E$; we prove that $\|Tx\| \geq \|x\|$. Let $f \in T^d(B_{F'})$. Then, with $g \in B_{F'}$ such that $f = T^d g$,

$$|f(x)| = |g(Tx)| \leq \|Tx\|.$$

This inequality then also holds for all $f \in \overline{T^d(B_{F'})}$, all $f \in (\overline{T^d(B_{F'})})^e = B_{E'}$. Then, by normpolarity,

$$\|x\| = \sup\{|f(x)| : f \in B_{E'}\} \leq \|Tx\|. \qquad \square$$

Remark. We see that the decomposition $T = T_2 \circ T_1$ (T_1 dense, T_2 isometry) leads to a similar decomposition $T^d = T_1^d \circ T_2^d$ in $\mathcal{C}_K$ because T_2^d is 'dense' in the sense that the smallest closed edged submodule of $B_{(\overline{TE})'}$ containing $T_2^d(B_{F'})$ is $B_{(\overline{TE})'}$ and T_1^d is an embedding.

As a corollary we obtain the following structure theorem for $\mathcal{C}_K$.

Theorem 5.2. *A topological B_K-module A is in $\mathcal{C}_K$ if and only if there is a set I and a subset L of $B_K^{\{I\}} := \{(\lambda_i)_{i \in I} \in B_K^I : \lim \lambda_i = 0\}$ such that A is isomorphic (as a topological B_K-module) to*

$$\{(\mu_i)_{i \in I} \in B_K^I : \textstyle\sum \lambda_i \mu_i = 0 \text{ for all } (\lambda_i)_{i \in I} \in L\}.$$

Proof. It suffices to consider the 'only if' part, so suppose $A \in \mathcal{C}_K$. By Theorem 4.6 we may assume $A = B_{E'}$ for some $E \in \mathcal{B}_K$. There is a K-Banach space M with an orthonormal base $\{e_i : i \in I\}$ and a quotient map $\pi : M \to E$ (e.g. take $I = B_E$, $M = \mathrm{co}\, I$ and $\pi((\lambda_i)_{i \in I}) = \sum \lambda_i . i$). Now $N := \mathrm{Ker}\, \pi$ has an orthonormal base by Gruson's theorem ([3], 5.9), say $\{f_j : j \in J\}$. Set $L := \{(\lambda_{ij})_{i \in I} : j \in J\}$, where, for each $j \in J$, $f_j = \sum_i \lambda_{ij} e_i$ is the expansion of f_j with respect to the base $\{e_i : i \in I\}$. Then $L \subset B_K^{(I)}$.

The map $h : f \mapsto (f(e_i))_{i \in I}$ is easily seen to be a homeomorphism $B_{M'} \to B_K^I$. By Theorem 5.1(i) the map $\pi^d : B_{E'} \to B_{M'}$ is a homeomorphism into. Now one verifies directly (using the fact that $\pi^d(B_{E'}) = \{f \in B_{M'} : f = 0 \text{ on } N\}$) that $h \circ \pi^d$ maps $B_{E'}$ onto $\{(\mu_i)_{i \in I} : B_K^I : \sum \lambda_i \mu_i = 0 \text{ for all } (\lambda_i)_{i \in I} \in L\}$. □

§6. SPACES OF COUNTABLE TYPE

We characterize spaces of countable type in terms of their associated compactoid.

Proposition 6.1. (Compare [7], 5.1.) *For a normpolar K-Banach space E the following are equivalent.*

(α) *E is of countable type.*

(β) *The weak star unit ball of E' is metrizable.*

Proof. (α) ⇒ (β). Choose $e_1, e_2, \ldots$ in B_E whose linear span is dense in E. The formula $\phi(f) = (f(e_1), f(e_2), \ldots)$ defines a homomorphism $B_{E'} \to B_K^{\mathbb{N}}$ which is easily seen to be a homeomorphism of $B_{E'}$ into the product space $B_K^{\mathbb{N}}$ which is metrizable hence so are $\varphi(B_{E'})$ and $B_{E'}$.

(β) ⇒ (α). Let $\lambda \in K$, $|\lambda| > 1$. By metrizability there exist ([6], 8.2) $f_1, f_2, \ldots \in \lambda B_{E'}$ with $w' - \lim_{n \to 0} f_n = 0$ such that

$$(*) \qquad B_{E'} \subset \overline{\mathrm{co}}\{f_1, f_2, \ldots\} \subset \lambda B_{E'}.$$

The formula $\phi(x) = (f_1(x), f_2(x), \ldots)$ defines a K-linear map $E \to c_0$. For each $x \in E$ we have

$$\|\phi(x)\| = \sup_n |f_n(x)| = \sup\{|g(x)| : g \in \overline{\mathrm{co}}\{f_1, f_2, \ldots\}\}$$

so that by normpolarness and $(*)$

$$\|x\| \leq \|\phi(x)\| \leq |\lambda|\, \|x\|.$$

Thus, E is linearly homeomorphic to a subspace of c_0 and therefore of countable type. □

§7. REFLEXIVITY

We prove that reflexivity of E is equivalent to automatic continuity of homomorphisms with domain $B_{E'}$:

Theorem 7.1. *For a compactoid $A \in \mathcal{C}_K$ the following are equivalent.*

(α) *For every $B \in C_K$ each module homomorphism $A \to B$ is continuous.*

(β) *For every absolutely convex compactoid subset B of a Hausdorff locally convex space over K each module homomorphism $A \to B$ is continuous.*

(γ) *There is no strictly stronger topology on A for which it is an embeddable compactoid.*

(δ) *A is isomorphic to the weak star unit ball of the dual of a reflexive space.*

(ε) *A is isomorphic to the weak unit ball of a reflexive space.*

Proof. $(\alpha) \Leftrightarrow (\beta)$ and $(\delta) \Leftrightarrow (\varepsilon)$ are almost obvious. We prove $(\beta) \Rightarrow (\gamma) \Rightarrow (\delta) \Rightarrow (\alpha)$. Suppose (β) and let τ be a stronger topology on A for which (A, τ) is an embeddable compactoid. Then the identity $A \to (A, \tau)$ is continuous by (β), hence τ is also weaker than the initial topology and we have (γ). Assume (γ). Let $A = B_{E'}$ for some Banach space $E \in \mathcal{B}_K$. The identity $(B_{E'}, \sigma(E', E'') \mid B_{E'}) \to (B_{E'}, \sigma(E', E) \mid B_{E'})$ is continuous hence a homeomorphism by (γ) so for each $\theta \in E''$ the set $\operatorname{Ker} \theta \cap B_{E'}$ is w'-closed. Then $\operatorname{Ker} \theta$ is w'-closed by [9], 3.1 and θ is w'-continuous. We see that $E'' = j_E(E)$; i.e. E is reflexive and (δ) is proved. Finally to show $(\delta) \Rightarrow (\alpha)$, let $A = B_{E'}$, $B = B_{X'}$ where $E, X \in \mathcal{B}_K$, let $\varphi : B_{E'} \to B_{X'}$ be any B_K-module homomorphism. Let $i \mapsto f_i$ be a net in $B_{E'}$ such that $f_i \to 0$ in w'. Then, by reflexivity, $f_i \to 0$ weakly. Now φ is norm continuous hence weakly continuous so $\varphi(f_i) \to 0$ weakly. Then certainly $\varphi(f_i) \to 0$ in w'. We see that φ is w'-continuous yielding (α). $\square$

Corollary 7.2. (i) *Let K be spherically complete. Then the only members A of C_K satisfying (α)–(ε) are the finite dimensional ones.*

(ii) *Let K be not spherically complete. Then every metrizable $A \in C_K$ satisfies (α)–(ε).*

Proof. (i) Reflexive spaces are finite-dimensional ([3], 4.16).

(ii) Every Banach space of countable type is reflexive ([3], 4.18). $\square$

§ 8. ORTHOGONAL BASES

We characterize the spaces in $\mathcal{B}_K$ with an orthogonal base in terms of the dual category C_K.

Theorem 8.1. *For a normpolar K-Banach space the following are equivalent.*

(α) *E has an orthogonal base.*

(β) *$B_{E'}$ is a product of edged bounded discs in K.*

Proof. $(\alpha) \Rightarrow (\beta)$. Let $\{e_i : i \in I\}$ be an orthogonal base of E. The formula

$$\phi(f) = (f(e_i))_{i \in I}$$

defines a homomorphism $B_{E'} \to \prod_{i \in I} C_i$ where $C_i := \{\lambda \in K : |\lambda| \leq \|e_i\|\}$. It is easily seen to be a homeomorphism into. To prove surjectivity let $\eta_i \in C_i$ for each $i \in I$. The formula

$$f(\sum \lambda_i e_i) = \sum \lambda_i \eta_i \quad (\lambda_i \in K,\ \|\lambda_i e_i\| \to 0)$$

defines a linear map $E \to K$. We have, with $x = \sum \lambda_i e_i$,

$$|f(x)| \leq \sup_i |\lambda_i n_i| \leq \sup_i |\lambda_i|\,\|e_i\| = \|x\|.$$

It follows that $f \in B_{E'}$ and that $\phi(f) = (\eta_i)_{i \in I}$.

$(\beta) \Rightarrow (\alpha)$. Suppose $B_{E'} = \prod_{i \in I} C_i$ where, for each $i \in I$, $C_i = \{\lambda \in K : |\lambda| \leq r_i\}$. We identify E to $\mathrm{Hom}(B_{E'}, K)$ with the sup norm (Remark 4 following 4.6). For each $j \in I$ let $e_j \in E$ be the j^{th} coordinate function $\prod_{i \in I} C_i \to K$ (then $\|e_i\| = r_i$ for each i) and let $\pi_j : \prod_{i \in I} C_i \to \prod_{i \in I} C_i$ be the map defined by

$$(\pi_j(c))_i = \begin{cases} c_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

For each $x \in E$ there exist $\xi_i \in K$ such that $x \circ \pi_i = \xi_i e_i$ for each $i \in I$. Straightforward verifications show that $(e_i)_{i \in I}$ is orthogonal and that $x = \sum_{i \in I} \xi_i e_i$ with respect to the sup norm. Hence E has an orthogonal base. □

Corollary 8.2. *For a normpolar K-Banach space the following are equivalent.*

(α) *E has an orthonormal base*

(β) *$B_{E'}$ is a power of B_K.*

Proof. It suffices to read the previous proof for the special case where $\|e_i\| = 1$ and $r_i = 1$ for all i. □

§ 9. STABILITY PROPERTIES

In this section we discuss some 'convexified versions' of compactness properties and consider their counterparts in Banach space theory. Let us call an $A \in \mathcal{C}_K$ *monocompact* if for each $B \in \mathcal{C}_K$, each injective $\varphi \in \mathrm{Hom}(A, B)$ is a homeomorphism into. We quote the next result from [1] and [4] (see also the remark following 9.2).

Proposition 9.1. *Let $A \in \mathcal{C}_K$. If K is spherically complete or A is metrizable then A is monocompact.*

Proposition 9.2. *For a Banach space $E \in \mathcal{B}_K$ the following are equivalent.*

(α) *Each weakly dense subspace of E is norm dense.*

(β) *$B_{E'}$ is monocompact.*

Proof. $(\alpha) \Rightarrow (\beta)$. Let $\varphi \in \mathrm{Hom}(B_{E'}, X)$ be injective where $X \in \mathcal{C}_K$. Then X has the form $B_{F'}$ for some $F \in \mathcal{B}_K$; let us denote φ^d (see 4.7) by $T : F \to E$. Injectivity of φ implies that $\overline{TF}^w = E$ (if $f \in B_{E'}$, $f = 0$ on TF then $\varphi(f) = T^d(f) = f \circ T = 0$, hence $f = 0$). By assumption TF is norm dense so that φ is a homeomorphism into by Theorem 5.1.

$(\beta) \Rightarrow (\alpha)$. Suppose F is a weakly dense but norm closed subspace of E with inclusion map $T : F \to E$. Then T', hence T^d, is injective. By (β), T^d is a homeomorphism into, by Theorem 5.1 F is norm dense in E. □

Remark. Let $E \in \mathcal{B}_K$. If K is spherically complete or E is of countable type we have (α) and this furnishes a new proof of Proposition 9.1.

Next, we ask when a $\varphi \in \mathrm{Hom}(A,B)$ $(A, B \in \mathcal{C}_K)$ is an open mapping $A \to \varphi(A)$. We need two lemmas.

Lemma 9.3. *Let D be a closed subspace of a Banach space $E \in \mathcal{B}_K$ with inclusion map $T : D \to E$. Suppose T^d is an open mapping $B_{E'} \to T^d B_{E'}$. Then for each $a \in E$ there is a finite-dimensional subspace F of D such that* $\mathrm{dist}(a,D) = \mathrm{dist}(a,F)$.

Proof. I. We may assume that the valuation of K is dense. Suppose the conclusion were false. Then there exists an $a \in E$ such that $\mathrm{dist}(a,D) < \mathrm{dist}(a,F)$ for each finite-dimensional space $F \subset D$. Set $\varepsilon := \mathrm{dist}(a,D)$.
II. Consider the obvious decomposition

$$\begin{tikzcd} B_{E'} \arrow[rr, "T^d"] \arrow[dr, "\pi"'] & & T^d B_{E'} \\ & S := B_{E'}/\mathrm{Ker}\, T^d \arrow[ur, "\varphi"'] & \end{tikzcd}$$

where the B_K-module S carries the quotient topology. Then φ is a homeomorphism by assumption. The seminorm $f \mapsto |f(a)|$ is w'-continuous on E' hence on $B_{E'}$. Therefore the quotient 'seminorm' q on S defined by

$$q : \pi(f) \mapsto \inf\{|f(a) - h(a)| : h \in \mathrm{Ker}\, \pi\}$$

is continuous. Then $q \circ \varphi^{-1}$ is continuous on $T^d B_{E'}$ so that $\{h \in T^d B_{E'} : (q \circ \varphi^{-1})(h) < \varepsilon\}$ is an open zero neighbourhood and therefore is the intersection of $T^d B_{E'}$ and a w'-zero neighbourhood in D'. In other words, there exists a finite set $X \subset D$ such that

$$(*) \qquad h \in T^d B_{E'}, |h| \leq 1 \text{ on } X \to (q \circ \varphi^{-1})(h) < \varepsilon.$$

III. By the assumption of I there is a $c \in K$ with

$$\mathrm{dist}(a,D) < |c| < \mathrm{dist}(a,[X]).$$

The map $\lambda a + v \mapsto \lambda c$ $(\lambda \in K, v \in [X])$ is easily seen to be an element of $(Ka + [X])'$ with norm $\leq |c|/\mathrm{dist}(a,[X]) < 1$. By normpolarity it can be extended to an $f \in E'$ whose norm is < 1. Choose a $\xi \in K$ with $|\xi| \geq 1$, $\|\xi f\| < 1$. Then $\xi f \in B_{E'}$, $T^d(\xi f) = 0$ on X so from $(*)$ we obtain $(q \circ \varphi^{-1})T^d(\xi f) < \varepsilon$ i.e. $q(\pi(\xi f)) < \varepsilon$ (see diagram), so there exists an $h \in \mathrm{Ker}\, \pi$ with $|\xi f(a) - h(a)| < \varepsilon$. Now we have $|\xi f(a)| = |\xi|\,|c| > \mathrm{dist}(a,D) = \varepsilon$ while $h \in \mathrm{Ker}\, \pi$ implies

$h \in B_{E'}$, so $\|h\| \le 1$ and $h = 0$ on D. Then $|h(a)| = \inf\{|h(a-d)| : d \in D\} \le \mathrm{dist}(a, D) = \varepsilon$. Then $|\xi f(a) - h(a)| = \max(|\xi f(a)|, |h(a)|) = \varepsilon$, a contradiction. □

Lemma 9.4. *Let E be a Banach space over a densely valued field K with the following property $(*)$.*

$$(*) \qquad \begin{cases} \text{For each } a \in E \text{ and each closed subspace } D \subset E \text{ there is a finite-} \\ \text{dimensional space } F \subset D \text{ such that } \mathrm{dist}(a, D) = \mathrm{dist}(a, F). \end{cases}$$

Then E is finite-dimensional.

Proof. Direct verifications show that $(*)$ is stable for the forming of closed subspaces and quotients by closed subspaces. So suppose there exists an infinite dimensional space E with $(*)$. Then there exists one of countable type E_1. Now c_0 is a quotient of E_1 ([2], 3.1) hence c_0 has $(*)$. But every finite-dimensional subspace of c_0 has an orthogonal base so, for every finite-dimensional subspace F of c_0, $\mathrm{dist}(a, F)$ is attained for every $a \in c_0$. Then $(*)$ implies that $\mathrm{dist}(a, D)$ is attained for every closed subspace D of c_0. But this is impossible (let D be the closed linear span of a maximal orthogonal set that is not an orthogonal base, see [3], p. 73). □

The next corollary obtains.

Theorem 9.5. *For an $A \in C_K$ the following are equivalent.*

(α) *For all $B \in C_K$, each $\varphi \in \mathrm{Hom}(A, B)$ is an open mapping $A \to \varphi(A)$.*

(β) *The valuation of K is discrete or A is finite-dimensional.*

Proof. For $(\beta) \Rightarrow (\alpha)$ see [10], § 3. To prove $(\alpha) \Rightarrow (\beta)$ let $A = B_{E'}$ where $E \in B_K$, let D be a closed subspace. Then by (α) the adjoint T^d of the inclusion map $T : D \to E$ is an open mapping $B_{E'} \to T^d B_{E'}$. By Lemmas 9.3 and 9.4 E is finite-dimensional hence so are E' and A. □

Looking at Theorem 9.5 one might wonder if $\varphi(A)$ is in C_K for all $\varphi \in \mathrm{Hom}(A, B)$ $(B \in C_K)$. Let us call an $A \in C_K$ *strictly epicompact* (see the remark below) if that conclusion holds.

Theorem 9.6. *Let $A \in C_K$.*

(i) *If K is spherically complete then A is strictly epicompact.*

(ii) *If K is not spherically complete and A is strictly epicompact then* $\dim A < \infty$.

Proof. (i) Let $B \in C_K$, $\varphi \in \mathrm{Hom}(A, B)$. Now $\varphi(A)$ is c-compact hence complete. See [11] for a proof of $\varphi(A)^e = \varphi(A)$.

(ii) For metrizable $A \in C_K$ this was proved in [2], 6.28; it is not hard to extend this result to any $A \in C$. □

Remark. If we relax the condition on $A \in C_K$ to just epicompactness (which means by definition that $\varphi(A)^e \in C_K$ for all $\varphi \in \mathrm{Hom}(A, B)$) then (ii) can be improved. For this and a study of an analogous modification of the notion of openness, see [12].

REFERENCES

1. Gruson, L. – Théorie de Fredholm p-adique. Bull. Soc. Math. France **94**, 67–95 (1966).
2. Rooij, A.C.M. van – Notes on p-adic Banach spaces. Reports 7633 and 7725, Mathematisch Instituut, Katholieke Universiteit, Nijmegen (1976, 1977).
3. Rooij, A.C.M. van – Non-archimedean functional analysis. Marcel Dekker, New York (1978).
4. Schikhof, W.H. – Topological stability of p-adic compactoids under continuous injections. Report 8644, Department of Mathematics, Catholic University, Nijmegen (1986).
5. Schikhof, W.H. – Compact-like sets in non-archimedean functional analysis. Proceedings of the Conference on p-adic analysis, Hengelhoef, 137–147 (1986).
6. Schikhof, W.H. – Locally convex spaces over nonspherically complete valued fields I, II. Bull. Soc. Math. Belg. Sér. B **38**, 187–224 (1986).
7. Schikhof, W.H. – A connection between p-adic Banach spaces and locally convex compactoids. Report 8736, Department of Mathematics, Catholic University, Nijmegen (1987).
8. Schikhof, W.H. – The continuous linear image of a p-adic compactoid. Proc. Kon. Ned. Akad. Wet. A **92**, 119–123 (1989).
9. Schikhof, W.H. – The p-adic Krein–Šmulian theorem. In: p-adic functional analysis, J.M. Bayod, N. de Grande-De Kimpe and J. Martínez-Maurica (eds.), Marcel Dekker, New York, 177–189 (1991).
10. Schikhof, W.H. – Zero sequences in p-adic compactoids. In: p-adic functional analysis, J.M. Bayod, N. de Grande-De Kimpe and J. Martínez-Maurica (eds), Marcel Dekker, New York, 227–236 (1991).
11. Schikhof, W.H. – More on duality between p-adic Banach spaces and compactoids. Report 9301, Department of Mathematics, Catholic University, Nijmegen (1993).
12. Schikhof, W.H. – The equalization of p-adic Banach spaces and compactoids. Report 9435, Department of Mathematics, University of Nijmegen (1994).
13. Springer, T.A. – Une notion de compacité dans la théorie des espaces vectoriels topologiques. Indag. Math. **27**, 182–190 (1965).